Conservation laws & Parabolic Monge-Ampère equations

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Theorem

If a 2nd-order, scalar, parabolic PDE,

$$\frac{\partial u}{\partial x^0} = F\left(x^0, x^i, u, \frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^i \partial x^j}\right) \quad \text{for} \quad i,j = 1, \ldots, n,$$

has at least one nontrivial conservation law, then it is in fact of Monge-Ampère type:

$$\frac{\partial u}{\partial x^0} = \sum_{\substack{\{0\}\subseteq I,J\subseteq\{0,\ldots,n\} \\ |I|=|J|}} A_{I,J}\left(x^0, x^i, u, \frac{\partial u}{\partial x^i}\right) H_{I,J},$$

where the $H_{I,J}$ are minor subdeterminants of the Hessian of u.

This Theorem has been known since the nineties for dimension $1+1$ (Bryant & Griffiths) and dimension $1+2$ (Clelland).

Definition

- An *exterior differential system* $(M, \mathcal{I})$ is a smooth manifold M and a graded, differentially closed ideal $\mathcal{I}$ in $\Omega^*(M)$.
- A submanifold $\iota\colon \Sigma \hookrightarrow M$ is an *integral submanifold* of $(M, \mathcal{I})$ if the pullback $\iota^*\mathcal{I}$ is identically zero.

- The idea is that $(M, \mathcal{I})$ is locally the same data as some PDE. Any solution to this PDE determines an integral submanifold of M.
- There is a canonical way to associate an exterior differential system to any reasonable PDE.

- For a parabolic equation

$$\frac{\partial u}{\partial x^0} = F\left(x^0, x^i, u, \frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^i \partial x^j}\right)$$

there is an associated exterior differential system

$$(M, \mathcal{I}) \subset (J^2(\mathbb{R}^{n+1}, \mathbb{R}), \mathcal{C}).$$

- Let me emphasize a point, which goes back to Cartan: $(M, \mathcal{I})$ is a geometric object, in the sense that M is a smooth manifold endowed with additional structure. In fact, since $\mathcal{I}$ is defined in terms of tensorial objects on M (in this case, differential forms), this geometry is "G-structural". (Caveat: sufficient constant rank of symmetry group.)
- The geometry of $(M, \mathcal{I})$ encodes information of the equation that is invariant under change of variables.
- One natural thing to do then is to consider coframings of M that are adapted to the geometry of $\mathcal{I}$.

Definition

A *(weakly) parabolic system* in $n+1$ variables is an exterior differential system $(M, \mathcal{I})$ with a spanning set of 1-forms $\theta_\varnothing, \theta_a, \omega^a, \pi_{ab} = \pi_{ba}$ for $a, b = 0, \dots n$ satisfying

1. The forms $\theta_\varnothing, \theta_a$ generate $\mathcal{I}$ as a differential ideal.
2. The structure equations

$$d\theta_\varnothing \equiv -\sum_{a=0}^{n} \theta_a \wedge \omega^a \pmod{\theta_\varnothing}$$

$$d\theta_a \equiv -\sum_{b=0}^{n} \pi_{ab} \wedge \omega^b \pmod{\theta_\varnothing, \theta_b} \qquad a, b = 0, \dots, n.$$

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3. The parabolic symbol relation

$$\sum_{i=1}^{n} \pi_{ii} \equiv 0 \pmod{\theta_\varnothing, \theta_a, \omega^a}.$$

- It follows from the definition that M has dimension $2n + 3 + \frac{(n+2)(n+1)}{2} - 1$, which is one less than the dimension of $J^2(\mathbb{R}^{n+1}, \mathbb{R})$. Indeed, the coframing means that M may be locally EDS-embedded as a hypersurface in $J^2(\mathbb{R}^{n+1}, \mathbb{R})$, thus defining a differential equation. This equation will have parabolic symbol.
- If M has a coframing as in the definition, there are many such coframings, a G-structure's worth.
- After working out some geometry, you find that some parabolic systems are special! One way to be special is for $(M, \mathcal{I})$ to have a "de-prolongation" to a Monge-Ampère system.

Definition

An EDS $(M_{-1}, \mathcal{I}_{-1} = \{\theta_\varnothing, \Upsilon\})$ is a *quasi-parabolic Monge-Ampère system* with $n+1$ degrees of freedom if

1. M_{-1} is $2n+3$ dimensional.
2. $\theta_\varnothing$ is contact.
3. Υ is an $(n+1)$-form that is *once* degenerate: the equation

$$\alpha \wedge \Upsilon \equiv 0 \pmod{\theta_\varnothing,\ d\theta_\varnothing}$$

has a 2-dimensional space of solutions, spanned by $\theta_\varnothing$ and another 1-form ω^0.

- As the name suggests, these exterior differential systems are (locally) equivalent to Monge-Ampère equations. The definition of non-degenerate Monge-Ampère systems is due to Bryant, Griffitsh, & Grossman. This is just the obvious generalization.

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- The modifier *quasi* is necessary, because generic Monge-Ampère equations are non-linear, and thus may have differing symbol when linearized at different solutions.
- There are special Monge-Ampère equations that don't have this issue, and the geometry leads you inevitably to them.

- Morally, a point of M_{-1} is a 1-jet of a solution to associated parabolic PDE. But the symbol depends on 2nd order information.
- There is a process, *prolongation*, which takes an exterior differential system and formally adjoins higher derivatives.

$$\begin{array}{ccc} & & (M^{(1)}, \mathcal{I}^{(1)}) \\ & \overset{\exists !}{\nearrow} & \downarrow \\ \Sigma & \longrightarrow & (M, \mathcal{I}) \end{array}$$

- This process can be repeated as many times as you like, producing equivalent EDSs that see arbitrarily high order information.
- At the end of this process is the infinite prolongation $(M^{(\infty)}, \mathcal{I}^{(\infty)})$, which is an infinite dimensional manifold, but not unreasonable.

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1. M_{-1} is $2n+3$ dimensional.
2. $\theta_\varnothing$ is contact.
3. Υ is an $(n+1)$-form that is *once* degenerate: the equation

$$\alpha \wedge \Upsilon \equiv 0 \quad (\text{mod } \theta_\varnothing,\ d\theta_\varnothing)$$

has a 2-dimensional space of solutions, spanned by $\theta_\varnothing$ and another 1-form ω^0.

- The prolongation $M_{-1}^{(1)}$ of M_{-1} does have well defined symbol at each point, although it may be discontinuous. Due to the one-degeneracy of M_{-1}, the symbol at any point of $M_{-1}^{(1)}$ will have at least one zero. Generally, there will be an open set in each fiber of $M_{-1}^{(1)} \to M_{-1}$ where the symbol is honestly parabolic.

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   has a 2-dimensional space of solutions, spanned by $\theta_\varnothing$ and another 1-form ω^0.

- So, given a parabolic system $(M, \mathcal{I})$, it is fair to ask if M is contained in the prolongation of some quasi-parabolic Monge-Ampère system. In this case, M is locally equivalent to a Monge-Ampère equation.

When is a parabolic system Monge-Ampère?

The effective answer: there are local invariants (the Monge-Ampère invariants) that one derives from the equivalence problem. These are computable in any given example and provide a direct test.

When is a parabolic system Monge-Ampère?

Proposition (M. 2020)

A parabolic system $(M, \mathcal{I})$ is Monge-Ampère if and only if there is an $(n+1)$-form Υ on M,

$$\Upsilon \equiv \sum_{i=1}^{n} \theta_i \wedge \omega_{(i)} + (\text{ terms quadratic in } \mathcal{I}) \pmod{\theta_\varnothing}$$

such that

$$d\Upsilon \equiv 0 \pmod{\theta_\varnothing,\ d\theta_\varnothing}$$

- Here the ommited index notation $\omega_{(i)}$ denotes the product of ω^0 through ω^n exclusive of ω^i (technically, up to sign)

Definition

For $(M, \mathcal{I})$ a parabolic system of $n+1$ degrees of freedom, a *conservation law* of M is an n-form φ on $M^{(\infty)}$ for which $d\varphi \in \mathcal{I}^{(\infty)}$, up to equivalence by closed n-forms and ones already in $\mathcal{I}^{(\infty)}$.

- Suppose $\varphi \notin \mathcal{I}^{(\infty)}$ and not closed, but $d\varphi \in \mathcal{I}^{(\infty)}$. Then φ pulls back to any integral submanifold to a closed form.
- Such a form provides boundary value obstructions. For example, if Σ_- is an n-dimensional solution and $\int_{\Sigma_-} \varphi \neq 0$, then there is no $(n+1)$-dimensional solution extending Σ_-. Indeed,

$$0 \neq \int_{\Sigma_-} \varphi = \int_{\partial \Sigma} \varphi = \int_{\Sigma} d\varphi = 0.$$

- If we work in a contractible neighborhood of M, there is an alternative definition of conservation laws

Definition

For $(M, \mathcal{I})$ a parabolic system of $n+1$ degrees of freedom, a *differentiated conservation law* of M is a closed $(n+1)$-form $\Phi \in \mathcal{I}^{(\infty)}$.

- The equivalence comes from the short exact sequence

$$0 \to \mathcal{I}^{(\infty)} \to \Omega^*(M^{(\infty)}) \to \Omega^*(M^{(\infty)})/\mathcal{I}^{(\infty)} \to 0.$$

Theorem (M. 2016)

Any differentiated conservation law of an evolutionary parabolic system $(M, \mathcal{I})$ *is a closed* $(n+1)$*-form on* M *that has* $\mathcal{I}$*-linear part*

$$\Phi \equiv A \sum_{i=1}^{n} \theta_i \wedge \omega_{(i)} \quad \left(mod\ \theta_\varnothing, \Lambda^2 \mathcal{I}^{(\infty)} \right).$$

Furthermore, A satisfies an auxilliary differential equation that depends only on the local invariants of M*.*

- A key point of this Theorem is that Φ descends to M, as opposed to $M^{(\infty)}$.
- This is proved using the Characteristic Cohomology theory developed by Bryant & Griffiths. Some spectral sequences reduce this to a calculation of the Spencer complex of the (parabolic) symbol of M, plus a bit more to account for the non-vanishing subprincipal symbol.

Theorem (M. 2020)

If a parabolic system $(M, \mathcal{I})$ has at least one non-trivial conservation law, then it is of Monge-Ampère type.

Proof.

Let

$$\Phi \equiv A \sum \theta_i \wedge \omega_{(i)} + \dots \pmod{\theta_\varnothing}$$

be a conservation law of M, with A a nowhere zero function on M. Since A is defined on M, there is an admissable change of coframing that absorbs A, so without loss of generality,

$$\Phi \equiv \sum \theta_i \wedge \omega_{(i)} + \dots \pmod{\theta_\varnothing}.$$

As a conservation law, Φ is closed, so

$$d\Phi \equiv 0 \pmod{\theta_\varnothing,\ d\theta_\varnothing}.$$

□

Thanks for listening!

- Conservation laws are useful for understanding the local invariants, even when they don't exist!
- From the structure equations of a parabolic system, there are 1-forms ξ_i^0 and ξ_i^j such that

$$d\theta_i \equiv \ldots - \xi_i^0 \wedge \theta_0 - \xi_i^j \wedge \theta_j - \pi_{ia} \wedge \omega^a \pmod{\theta_\varnothing}.$$

- There are functions V_i^{abc}, with some symmetries so that

$$\xi_i^a \equiv V_i^{abc} \pi_{bc} \pmod{\theta_\varnothing, \theta_a, \omega^a},$$

  and these contain the invariants that determine if M is Monge-Ampère.
- If they all vanish, life is easy, and M is a special type, parabolic Monge-Ampère.
- However, if certain of them vanish, then M is a general quasi-parabolic Monge-Ampère. But the direct proof of this would be very messy! On the other hand, it's a simple corollary of the characteristic cohomology calculations of conservation laws.